Math 210C Lecture 3 Notes

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April 5, 2019

1 Isolated and Embedded Primes, Algebraic Sets, and the Zariski Topology

1.1 Isolated and embedded primes

Last time, we showed that if R is noetherian every proper ideal is decomposable. We also showed that if I is decomposable, then the associated primes of I are unique, independent of the choice of decomposition.

Definition 1.1. If $I \subseteq R$ is proper, an **isolated prime** of I is a minimal element in the set of prime ideals containing I.

Last time, we gave a definition that depended on the decomposition. These are the same.

Proposition 1.1. *Let $I \subseteq R$ be decomposable. A prime ideal $\mathfrak{p}$ of R is an isolated prime of I if and only if it is minimal (under inclusion) among the associated primes in some primary decomposition of I.*

Proof. Let $I = \bigcap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}$-primary. Suppose $\mathfrak{p} \supseteq I$ is prime. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supseteq \sqrt{I} = \bigcap_{i=1}^n \sqrt{\mathfrak{q}_i} = \bigcap_{i=1}^n \mathfrak{p}_i$. By the lemma from last time, $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. This $\mathfrak{p}_i$ may not be minimal, but it contains a minimal prime. $\square$

Definition 1.2. An **embedded prime** is an associated prime that is not isolated.

Example 1.1. In $F[x, y]$, let $I = (xy, y^2) = (x, y)^2 \cap (y) = (x, y^2) \cap (y)$. The associated primes are $(x, y), (y)$. The only isolated prime is (y), and the only embedded prime is (x, y).

Proposition 1.2. *Let I be decomposable with $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ distinct isolated primes of I. Let Q be the minimal primary decomposition of I: $\mathfrak{q}_i \in Q$ is $\mathfrak{p}_i$ primary for $1 \leq i \leq n$. The ideal $\bigcap_{i=1}^n \mathfrak{q}_i$ is independent of the choice of Q.*

Proof. The idea is to localize at $S = R \setminus \bigcup_{i=1}^n \mathfrak{p}_i$. $\square$

Corollary 1.1. *Let $\mathfrak{p}$ be an isolated prime of I (decomposable). Then the unique $\mathfrak{p}$-primary ideal $\mathfrak{q}$ in a minimal decomposition of I is independent of the decomposition.*

Corollary 1.2. *The primary ideals in a noetherian ring R are exactly the irreducible ideals.*

Proof. Irreducible ideals are primary, so let $\mathfrak{p}$ be a primary ideal. It has a decomposition (namely itself), so it is an isolated prime of itself. So it is unique. $\square$

1.2 Algebraic sets and the Zariski topology

Let K be an algebraically closed field. Let $n \geq 1$, and let $R = K[x_1, \ldots, x_n]$. This is a free algebra of K, so if we have a finite dimensional algebra over K, then it is a quotient of R by some ideal. So studying R is important.

Definition 1.3. Let $S \subseteq K[x_1, \ldots, x_n]$. The **zero set** or **vanishing locus** of S is

$$V(S) = \{(a_1, \ldots, a_n) \in K^n : f(a_1, \ldots, a_n) = 0 \, \forall f \in S\}.$$

An **algebraic set** in K^n is a zero set of some S.

Remark 1.1. If $S = \{f_1, \ldots, f_n\}$, we write $V(S) = V(f_1, \ldots, f_n)$.

Remark 1.2. If $S \subseteq T$, then $V(T) \subseteq V(S)$.

Remark 1.3. $V(S) = V((S))$.

Remark 1.4. $V(S \cup T) = V(S) \cap V(T)$.

Example 1.2. Let $S = \{xy, y^2\} \subseteq K[x, y]$. xy vanishes when $x = 0$ or $y = 0$, and y^2 vanishes when $y = 0$. So $V(S) = \{(x, 0) : x \in K\}$.

Example 1.3. Look at $V(x - y, x^2 + y^2 - 1) \subseteq \mathbb{C}^2$. This is $V(x - y) \cap V(x^2 + y^2 - 1)$. So $x = y$, $2x^2 = 1$, and so $x = \pm\sqrt{1/2} = y$. So $V(x - y, x^2 + y^2 - 1) = \{\pm(1/\sqrt{2}, 1/\sqrt{2})\}$.

Proposition 1.3. *Algebraic sets have the following closure properties:*

1. *The intersection of any collection of algebraic sets is algebraic.*
2. *The union of any finite collection of algebraic sets is algebraic.*

Proof. For the first statement, $\bigcap_{i \in I} V(S_i) = V(\bigcup_{i \in I} S_i)$.

For the second statement, let $S, T \subseteq R$ with $I = (S)$ and $J = (T)$. Then $V(S) \cup V(T) = V(I) \cup V(J) \subseteq V(I \cap J)$. If $a \in V(I \cap J)$ and $a \notin V(I)$, then there exists $f \in I$ such that $f(a) \neq 0$. For any $g \in J$, $fg \in IJ \subseteq I \cap J$, so $(fg)(a) = 0$. So $g(a) = 0$, and since g was arbitrary, $a \in V(J)$. $\square$

Remark 1.5. $V(\varnothing) = K^n$, and $V(R) = V(1) = \varnothing$.

Definition 1.4. The **Zariski topology** on K^n is the topology $\{K^n \setminus V(S) : S \subseteq R\}$, i.e. the topology with closed sets the algebraic subsets of K^n. **Affine n-space** $\mathbb{A}^n_K$ is K^n endowed with this topology.

Proposition 1.4. *The Zariski topology is T_1; i.e. points are closed.*

Proof. Let $a = (a_1, \ldots, a_n) \in K^n$. Then let $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ be a maximal ideal of R. Then $V(\mathfrak{m}) = \{a\}$, so $\{a\}$ is algebraic. $\square$

Is the Zariski topolgoy Hausdorff?

Example 1.4. Let $n = 1$ and $f \in K[x]$¿ Then $V(f)$ is finite (roots of f). The topology on $\mathbb{A}^1_K$ is the cofinite topology (closed sets are $\mathbb{A}^1_K$ or finite sets). This is not Hausdorff.

Definition 1.5. Let $Z \subseteq \mathbb{A}^n_K$. $I(Z) = \{f \in R : f(a) = 0\, \forall a \in Z\}$ is an ideal of R.[1]

Remark 1.6. Let $f \in R$ with $f^k \in I(Z)$. Then $f(a)^k = 0$ for all $a \in Z$, so $f(a) = 0$ for all $a \in Z$. This means that $f \in I(Z)$. So $I(Z) = \sqrt{I(Z)}$ is a radical ideal.

Example 1.5. Let $a = (a_1, \ldots, a_n) \in \mathbb{A}^n_K$. Then $I(\{a\}) = (x_1 - a_1, \ldots, x_n - a_n)$.

Next time, we will prove the following theorem.

Theorem 1.1 (weak Nullstellensatz)**.** *Every maximal ideal of R has the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $(a_1, \ldots, a_n) \in \mathbb{A}^n_K$.*

[1] There is not a standard name for $I(Z)$.